Name:__________________

AB Calculus
4.1 Implicit Differentiation

Explicit form : $y = 3x^2 - 5$ (y on one side, x's on the other)

Implicit form : $x^2 - 2y^3 - 4y = 2$ (x's +y's all over, can't solve for y ! Don't try !)

Implicit Differentiation Process

1. Differentiate BOTH sides of equation with respect to x
2. Collect all terms involving $\frac{dy}{dx}$ on left side of equation, move all other terms to right side.
3. Factor out $\frac{dy}{dx}$ on left side
4. Solve for $\frac{dy}{dx}$ by dividing by other factor on left

Example Find $\frac{dy}{dx}$ for $x^3 - 3x^2y + 2xy^2 = 12$

$$x^3 - 3x^2y + 2xy^2 = 12$$

$$\frac{d}{dx}\left[x^3 - 3x^2y + 2xy^2\right] = \frac{d}{dx}(12)$$

$$3x^2 - 3\left(x^2\frac{dy}{dx} + y(2x)\right) + 2\left(x \cdot 2y\frac{dy}{dx} + y^2(1)\right) = 0$$

$$3x^2 - 3x^2\frac{dy}{dx} - 6xy + 4xy\frac{dy}{dx} + 2y^2 = 0$$

$$4xy\frac{dy}{dx} - 3x^2\frac{dy}{dx} = 6xy - 3x^2 - 2y^2$$

$$\frac{dy}{dx}(4xy - 3x^2) = 6xy - 3x^2 - 2y^2$$

$$\boxed{\frac{dy}{dx} = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}}$$

It doesn't have to be pretty (it just has to be right!)

Pitfalls with Implicit Differentiation

Differentiating "with respect to x"

a) $\frac{d}{dx}\left[x^3\right]=3x^2$ variables agree → simple power rule

↖ ↖

Variables Agree

b) $\frac{d}{dx}\left[y^3\right]=3y^2\frac{dy}{dx}$ variables disagree → chain rule

↖ ↖

Variables Disagree

c) $\frac{d}{dx}\left[x+3y\right]=1+3\frac{dy}{dx}$ chain rule $\frac{d}{dx}(3y)=3\cdot\frac{dy}{dx}$

d) $\frac{d}{dx}\left[xy\right]=$

$x\cdot\frac{d}{dx}\left[y\right]+y\cdot\frac{d}{dx}\left[x\right]$ Product Rule w/chain rule (for y)

$=x\frac{dy}{dx}+y$

You might find it easier to use y' instead of $\frac{dy}{dx}$ after you get the hang of it.

Implicit Differentiation
Example
2^{nd} Derivative

Find y'' for $x^2 + y^2 = 16$

1. First find y'

$$2x + 2yy' = 0$$
$$2yy' = -2$$
$$y' = \frac{2x}{2y} \qquad \leftarrow \textit{Simplify before moving onto } y''$$
$$y' = -\frac{x}{y} \qquad \swarrow$$

2. Take second derivative, remember chain rule:

$$y' = \frac{-x}{y}$$
$$y'' = -\left[\frac{y - xy'}{y^2}\right]$$

3. Sub in y'

$$y'' = -\left[\frac{y - x(-\frac{x}{y})}{y^2}\right]$$
$$= -\left[\frac{y + \frac{x^2}{y}}{y^2}\right]$$
$$= -\left[\frac{y + \frac{x^2}{y}}{y^2}\right]\left(\frac{y}{y}\right) = -\frac{y^2 + x^2}{y^3}$$

$WOW!$ $x^2 + y^2 = 16$

$\searrow$

$$y'' = -\frac{x^2 + y^2}{y^3} = \boxed{\frac{16}{y^3} = y''}$$

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AB CALCULUS
4.1 Implicit Differentiation

I. Find y'

1. $x^2 + y^2 = 25$

2. $x^2 y + 3xy^3 - x = 3$

3. $\sin(x^2 y^2) = x$

II. Find slope of tangent line <u>and</u> the equation of the tangent line at the given point

4. $x^2y - 5xy^2 + 6 = 0$ (3,1)

5. $\sin(xy) = y$ $(\frac{\pi}{2}, 1)$

6. $x^3y + y^3x = 10$ (1,2)

III. Find $\frac{d^2y}{dx^2}$

7. $x^2 + y^2 = 100$

8. $3x^2 - 4y^2 = 7$

9. $x^3 + y^3 = 1$

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ANSWERS TO IMPLICIT DIFFERENTIATION WORKSHEET

I. 1. $y' = -\frac{x}{y}$

2. $$y' = \frac{1-2xy-3y^3}{x^2+9xy^2}$$

3. $$y' = \frac{1-2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$$

II. 4. $$y' = \frac{5y^2-2xy}{x^2-10xy}$$

$$\left.\frac{dy}{dx}\right|_{(3,1)} = \frac{1}{21}$$

$$y-1 = \frac{1}{21}(x-3)$$

5. $$y' = \frac{y\cos(xy)}{1-x\cos(xy)}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2},1\right)} = 0$$

therefore

$$y = 1$$

6. $$y' = -\frac{3x^2y + y^3}{x^3 + 3xy^2}$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{14}{13}$$

$$y - 2 = -\frac{14}{13}(x - 1)$$

III. 7. $$y'' = \frac{-100}{y^3} \qquad \left(y' = -\frac{x}{y}\right)$$

8. $$y'' = -\frac{21}{16y^3} \qquad \left(y' = \frac{3x}{4y}\right)$$

9. $$y'' = -\frac{2x}{y^5} \qquad \left(y' = -\frac{x^2}{y^2}\right)$$